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A Pressure-Based Composite Grid Method for the Navier–Stokes Equations

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In this work, a pressure-based composite grid method is developed for solving the incompressible Navier–Stokes equations on domains composed of an arbitrary number of overlain grid blocks, where a conservative internal boundary scheme is devised to ensure that global conservation is maintained. Issues concerning the differences between the conservative internal boundary scheme developed for the pressure correction method with a staggered grid and that commonly used for density-based methods for compressible flow with nonstaggered grids are discussed. An organizational scheme is developed in order to provide a general and more flexible means for handling arbitrarily overlain grid blocks. Applications of the composite grid method to various model problems with complex geometry are used to illustrate the characteristics of the present procedure. © 1993 Academic Press, Inc.

1. INTRODUCTION

The numerical solution of the Navier–Stokes equations using finite difference methods requires the generation of a grid for the region of interest. For many problems of engineering interest, the generation of a single grid which discretizes the domain adequately for resolving the various flow features is very difficult or even impractical. This problem can be overcome to a limited extent by applying sophisticated grid generation schemes to construct a single grid with suitable characteristics; however, the degree of satisfaction achieved with such a process is highly problem dependent. An alternative approach is to partition the domain into a number of distinct blocks, each block being topologically simple. Grids can then be independently generated for each block with little difficulty. Furthermore, the grid resolution in each subdomain can be better controlled according to the fluid physics there.

For such a composite grid (or multiple-block) solution procedure, in which the governing equations are solved independently within each of the blocks, one of the main issues of importance concerns the transfer of information between the different blocks in the system. Strategies for transferring information between the blocks can be generally classified as either conservative or non-conservative. For many multiple-block solution techniques, information is transferred from block to block strictly via interpolation. This is the case for many multiple-block schemes using overlain grids [1–3], or abutting (patched) grids [4]. For some techniques employing the so-called Chimera grids, information transfer via interpolation is the only practical recourse, as the implementation of conservative transfer schemes becomes very difficult. For some applications, interpolation proves to be satisfactory; however, in many instances in which large solution gradients, or elliptic features such as recirculation exist in the vicinity of block boundaries, significant errors can be introduced into the solution. In addition, for schemes based upon a control volume formulation, the use of a non-conservative interface treatment, such as direct interpolation, may result in incompatible boundary conditions which can prevent the algorithm from converging, unless due attention is given to the choice of the interpolation scheme, which, in essence, makes the interface treatment consistent with the conservative concept. Hence, a conservative approach is preferable for transferring information between the grid blocks in the system. Much work has been done in this area, mainly in the context of density-based methods for compressible fluid flows. Rai [5–6], for example, has successfully developed and implemented conservative boundary schemes for Euler equation calculations on composite patched grids in a general curvilinear coordinate framework, for both explicit and implicit time integration schemes. Reggio *et al.* [7] have also developed conservative multiple-block strategies for the Euler equations, but using overlaid grids. In a recent work by Perng and Street [8], a composite grid technique was developed for solving the incompressible Navier–Stokes equations using a pressure-based method with the staggered grid arrangement; however, the interface treatment used for transferring information between the blocks is not conservative.

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In this paper, a multiple-block computational procedure is developed for solving the incompressible Navier–Stokes equations on domains comprised of an arbitrary number of overlapping grid blocks. A pressure correction algorithm in the spirit of SIMPLE [9] is used in conjunction with a staggered grid system to solve the continuity and momentum equations in a sequential fashion within each of the blocks of the domain, and a conservative interface treatment is used for transferring information between the blocks. There are a number of merits gained from using a sequential solution technique like SIMPLE along with a staggered grid. Sequential solution techniques allow one to accommodate a different number of equations depending on the physics involved, without the need for reformulating the solution algorithm. In addition, the basic algorithm can be extended in a unified framework to handle the various flow regimes, from incompressible to hypersonic [10–11]. Some of the merits of the staggered grid include the compactness of the discretization formulae and the elimination of artificial pressure boundary conditions. While the use of a staggered grid increases the complexity of the numerical scheme, we believe the basic merits of the staggered grid, mentioned above, outweight the drawbacks. With the use of a pressure correction method along with a staggered grid, new issues arise concerning the implementation of a conservative block boundary scheme for transferring information between the blocks of the system. The present work serves as a first step in illuminating the general issues involved in the development of a globally conservative multiple-block solver using a pressure correction scheme with the staggered grid system. The technique developed here can be incorporated in its entirety into a solution algorithm using curvilinear coordinates [12, 13].

2. GOVERNING EQUATIONS AND NUMERICAL ALGORITHM

The governing equations used here are the steady, incompressible two-dimensional Navier–Stokes equations, along with the corresponding form of the continuity equation, written in conservative form as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{1a}$$

$$\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho uv)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1b}$$

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right). \tag{1c}$$

According to SIMPLE [9], an initial guess is first made for the velocity and pressure field. The momentum equations are then solved to yield updated estimates for the velocity field. Next, a pressure correction equation is solved to give an updated estimate of the pressure field. The velocity field obtained above is also corrected using the pressure corrections to obtain a continuity satisfying velocity field at each iteration. If the continuity and momentum equations are not satisfied to the desired tolerance, then the process is repeated using the current estimates of the velocity and pressure fields. A staggered grid system, showing the physical locations of the two velocity components, pressure, and density associated with the grid point (i, j) is shown in Fig. 1.

Discretization of the Momentum Equations

Discretization of the momentum equations is accomplished using a control volume approach. Figure 2 shows a finite volume grid representation for a u control volume. The notation shown in the figure, and adopted here throughout, is standard for the SIMPLE algorithm with the staggered grid system. Upon integrating the u-momentum equation over the u control volume limits, one arrives at the general form of the conservation equation for u-momentum as

$$A_P u_P=\sum_{i=E,W,N,S} A_i u_i+(p_W-p_E)\,\Delta y. \tag{2}$$

The coefficients A_i and A_P represents the combined convection and diffusion fluxes associated with each of the nodes. In order to aid the discussion later needed for the development of the multiple-block algorithm, representative expres-

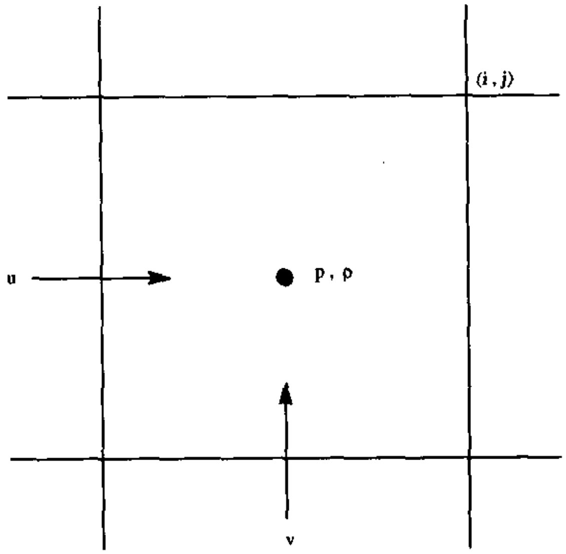

FIG. 1. Staggered grid system showing location of velocity components and scalar variables associated with grid point (i, j).

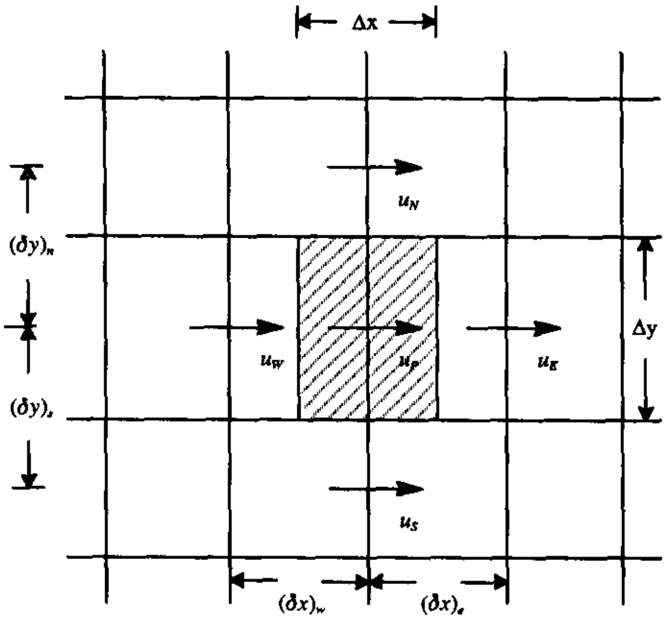

FIG. 2. *U* control volume used in deriving the discretized form of the *u*-momentum equation.

sions of these A_i terms are given, where the second-order central difference scheme is adopted for both the convection and diffusion terms, as

$$A_E = D_e(1 - 0.5\,|F_e/D_e| + \llbracket -F_e, 0.0 \rrbracket \tag{3a}$$

$$A_W = D_w(1 - 0.5\,|F_w/D_w|) + \llbracket F_w, 0.0 \rrbracket \tag{3b}$$

$$A_N = D_n(1 - 0.5\,|F_n/D_n|) + \llbracket -F_n, 0.0 \rrbracket \tag{3c}$$

$$A_S = D_s(1 - 0.5\,|F_s/D_s|) + \llbracket F_s, 0.0 \rrbracket \tag{3d}$$

$$A_P = A_E + A_W + A_N + A_S + (F_n - F_s + F_e - F_w). \tag{3e}$$

Equations (3) are part of a generalized form capable of representing several different discretization schemes, as reported in [14]. The F's and D's represent the convection and diffusion flux coefficients at the faces of the control volume, and are given by

$$F_e = (\rho u)_e\,\Delta y, \qquad F_w = (\rho u)_w\,\Delta y, \qquad F_n = (\rho v)_n\,\Delta x, \qquad F_s = (\rho v)_s\,\Delta x \tag{4a}$$

$$D_e = \frac{\mu_e\,\Delta y}{(\delta x)_e}, \qquad D_w = \frac{\mu_w\,\Delta y}{(\delta x)_w}, \qquad D_n = \frac{\mu_n\,\Delta x}{(\delta y)_n}, \qquad D_s = \frac{\mu_s\,\Delta x}{(\delta y)_s}. \tag{4b}$$

In the above expressions, $\llbracket a, b \rrbracket$ represents the maximum of the two arguments a and b. It is noted that the terms appearing within the parenthesis of Eq. (3e) are collectively equal to zero because of the mass continuity constraint. They are explicitly retained for helping the understanding of the interface treatment to be presented later. A similar procedure can be performed for the v-momentum equation by integrating the v-momentum equation over a v control volume, resulting in

$$A_P v_P = \sum_{i=E,W,N,S} A_i v_i + (p_S - p_N)\,\Delta x, \tag{5}$$

where the coefficients are identical to those previously given for the u-momentum equation.

Discretized Form of the Pressure Correction Equation

The continuity and momentum equations can be used to formulate a pressure correction equation. The pressure correction is used to update the pressure field, and in conjunction with the velocity correction formulas, to obtain a continuity satisfying velocity field at each iteration step [9]. In order to aid the discussion of the multiple-block procedure to be developed later, the discretized form of the pressure correction equation is presented below:

$$a_P p'_P = a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S + b \tag{6a}$$

$$a_E = \frac{\rho_e(\Delta y)^2}{A_{P_e}}, \qquad a_W = \frac{\rho_w(\Delta y)^2}{A_{P_w}}, \qquad a_N = \frac{\rho_n(\Delta x)^2}{A_{P_n}}, \qquad a_S = \frac{\rho_s(\Delta x)^2}{A_{P_s}} \tag{6b}$$

$$a_P = a_E + a_W + a_N + a_S \tag{6c}$$

$$b = [(\rho u^*)_w - (\rho u^*)_e]\,\Delta y + [(\rho v^*)_s - (\rho v^*)_n]\,\Delta x. \tag{6d}$$

In the above expressions, the starred velocity components represent those values obtained from the most recent solution of the momentum equations. The A_P terms are the coefficients A_P in the discretized forms of the momentum equations for the velocity components located on the faces of the pressure prime control volume.

3. COMPOSITE GRID CALCULATION PROCEDURE

As stated earlier, the goal of this study is to develop a computational procedure for solving the incompressible Navier–Stokes equations on domains composed from an arbitrary number of overlapping grid blocks, with each capable of having a user chosen discretization. In order to establish a basis for the following discussion, a representative example of such a composite grid is shown in Fig. 3. For the sake of developing the composite grid procedure, the Cartesian grid system is considered to simplify the

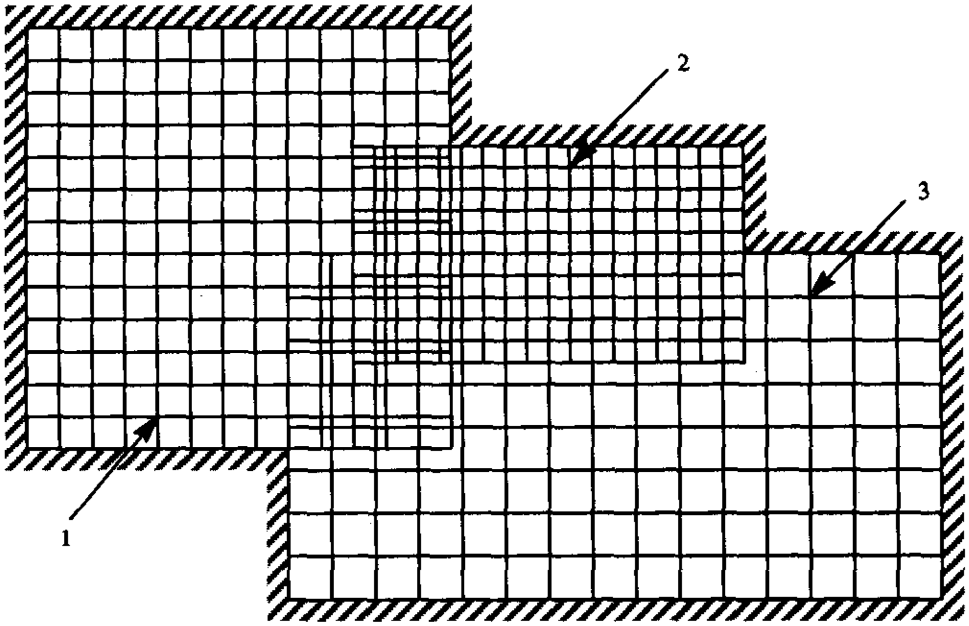

FIG. 3. Example of a composite grid composed of three overlapping grid blocks.

presentation. The ideas discussed below can be extended in their entirety to a curvilinear grid system. The development of an organizational methodology for handling arbitrarily constructed composite grids will be the first point of discussion. The global conservation conditions for the staggered grid are next discussed, followed by the development of the conservative internal boundary scheme.

Organization of Composite Grids

One of the primary difficulties in dealing with composite grids is the organizational task of determining the information flow from block to block. For a composite grid consisting of only two overlapping grid blocks, or for any composite grid in which no more than any two blocks in the system overlap at any single point in the domain, the channel of information flow is already determined, since there is only one neighboring block for each block in the system which can provide the required internal boundary data. However, for cases in which three, or even more grid blocks in the composite grid system overlap, the question of where internal boundary information is to be obtained becomes more difficult. In these cases, several grid blocks may be available to provide information across some parts of an internal boundary, while for other parts of the boundary, information may only be obtained from one other neighboring block. An example of this may be seen again from the simple composite grid system shown in Figure 3. Here, for the lower side of block two, which is entirely an internal boundary, for the right portion of this boundary, information can be obtained only from block three; however, for the left portion of this boundary, information may be obtained from either block one or block three.

It is apparent that the problem of handling composite grids consisting of multiple overlapping blocks, at least in terms of information transfer, is equivalent to determining what information flows across particular segments of each side of each block in the system. Each block side can have two types of segments associated with it, which we will designate as boundary condition segments and internal boundary segments. Boundary condition segments, which include both Dirichlet and outflow (i.e., gradient condition) segments are specified at the outset of the problem, and in terms of information flow, all required information is specified via the particular boundary condition for each segment. Internal boundary segments, on the other hand, are created by the overlappings of the blocks in the composite grid. Each side of each block in the system is spanned by a combination of one or more internal boundary segments and/or boundary condition segments. Across each of the internal boundary segments, information is obtained from a single neighboring grid block. While the internal boundary segments are essentially determined by the arrangement of the composite grid system, it should be noted that for composite grids containing regions in which more than two blocks overlap, each block may not have a unique set of internal boundary segments. This again relates to the fact that information may be obtained across portions of the internal boundaries from more than one of the neighboring blocks. Thus, some strategy must also be developed for selecting a set of internal boundary segments for blocks which have no unique set. Once the internal boundary segments for each side of each block in the composite grid system have been determined, then along with the boundary condition segments of the blocks, the complete paths of information flow into each block are specified, and the conservative internal boundary scheme can be implemented.

Our procedure for determining a unique set of internal boundary segments for the internal boundaries of each block in the composite grid system is as follows: First an intersection test is performed for each block in the composite grid with every other block to arrive at a preliminary set of internal boundary segments for that block. For general composite grids, the internal boundary segments for some of the internal boundaries may overlap along portions of the boundaries, resulting in an ambiguity in terms of information transfer through those portions of the boundaries. In order to resolve this ambiguity, each block in the composite grid system is given an overlapping priority, so that when two internal boundary segments overlap over a particular portion of an internal boundary, the segment with the highest priority takes precedence. In this way, a unique set of internal boundary segments for each internal boundary of each block in the composite grid can be obtained. This two-step process for uniquely determining the internal boundary segments for each block in the system is best illustrated by example.

Consider again, the composite grid system shown in Fig. 3. Let us determine a unique set of internal boundary segments for each internal boundary of block two in this three-block composite grid. For this block, the entire lower side is an internal boundary, part of the right side is an internal boundary, and the entire left side is an internal boundary. First, an intersection of block two independently with each of blocks one and three is performed. From these intersection tests, we obtain the preliminary internal boundary segments shown in Fig. 4a. The number by each of the segments indicates that segment was created by the intersection of block two with the block of that number. For the lower internal boundary as well as the left internal boundary of block two, the internal boundary segments overlap over a portion of the boundary, again, corresponding to the fact that, across these portions of the boundaries, information can be obtained from more than one block. Now, we further specify the following overlapping priorities for each block in the composite grid. Block number one is given the highest priority, followed by block two, and then block three. After using this priority information to eliminate internal boundary segment overlaps, we obtain the final set of internal boundary segments shown in Fig. 4b, for each internal boundary of block two. With this set of internal boundary segments, the information channels through the internal boundaries are precisely specified. Along with the boundary condition segments, a complete specification of the information flow into block two is obtained. It should be noted that this procedure for determining the exchange of information within a composite grid arrangement is quite general and can be applied to composite grids created from an arbitrary number of grid blocks overlaid in an arbitrary fashion.

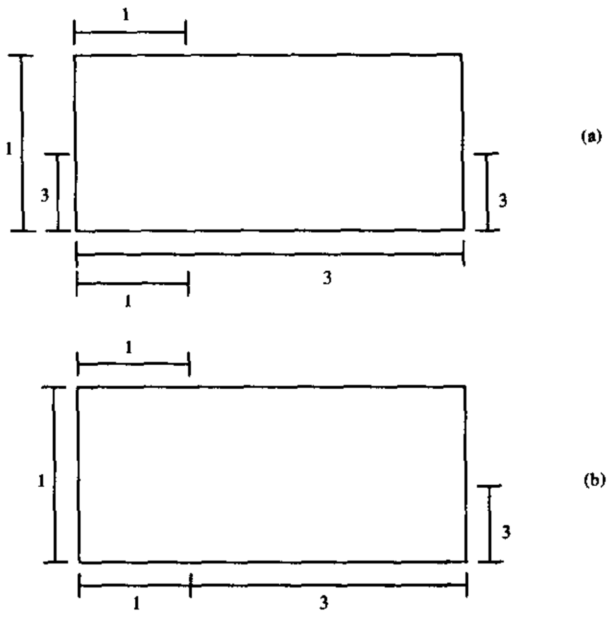

FIG. 4. Determination of internal boundary segments for block two in Fig. 3: (a) Preliminary internal boundary segments; (b) Final internal boundary segments.

Global Conservation Conditions for the Staggered Grid

Consider the u-momentum equation written in conservative form as

$$E_x + F_y = 0. \tag{7a}$$

Integrating this equation over a u control volume gives

$$(E_e - E_w)\,\Delta y + (F_n - F_s)\,\Delta x = 0, \tag{8}$$

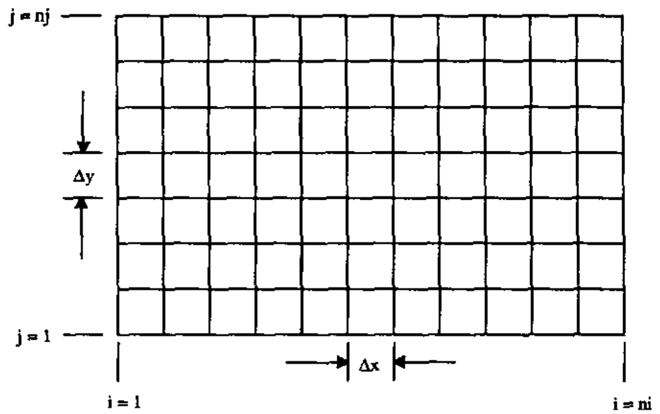

FIG. 5. Single block grid used in deriving global conservation condition for the u-momentum equation on the staggered grid.

where the terms $E_e\,\Delta y$ and $E_w\,\Delta y$ are respectively the total fluxes of u-momentum through the east and west control volume faces, and the terms $F_n\,\Delta x$ and $F_s\,\Delta x$ are respectively the total fluxes of u-momentum through the north and south control volume faces.

Now consider the single grid domain shown in Fig. 5 with uniform spacing in both coordinate directions. Summing the above equation for a u control volume over all the u control volumes in the domain yields

$$\begin{aligned} S &= \sum_{j=2}^{nj} \sum_{i=3}^{ni} \{(E_e - E_w)\,\Delta y + (F_n - F_s)\,\Delta x \\ &= \sum_{j=2}^{nj} (E_e|_{i=ni} - E_w|_{i=3})\,\Delta y \\ &\quad + \sum_{i=3}^{ni} (F_n|_{j=nj} - F_s|_{j=2})\,\Delta x = 0. \end{aligned} \tag{9}$$

Therefore, S is only a summation of the boundary fluxes, since the interior fluxes for neighboring control volumes cancel each other out. This equation represents the global conservation property for any control-volume based scheme as shown in Eq. (8). Note, however, for the staggered grid, the boundary in question is not the physical boundary of the domain as is usually the case when dealing with a nonstaggered grid, but the boundary formed by the boundary sides of the u control volumes along the physical boundary. This boundary will hereafter be referred to as the global conservation boundary. For the staggered grid system, three different global conservation boundaries exist, one for the continuity (pressure correction) equation, one for the u-momentum equation, and one for the v-momentum equation. The three global conservation boundaries for the staggered grid covered by the control volumes of the interior unknowns are shown in Fig. 6.

For composite grids, the global conservation property also requires that a summation of the u control volume equations over all the u control volumes in the composite grid results in only a summation of the u momentum fluxes across the boundary for the composite grid. Note that for such a summation, the total area of the summation should be equal to the area formed by an exclusive summation of the areas enclosed by the global conservation boundaries of each of the blocks forming the composite grid. By an exclusive summation, we mean that in an overlap zone created by the intersection of the global conservation boundaries of two blocks, the overlap area should only be included in the control volume summation of one of the two blocks.

For example, consider the composite grid shown in Fig. 7a, constructed from two blocks with a horizontal

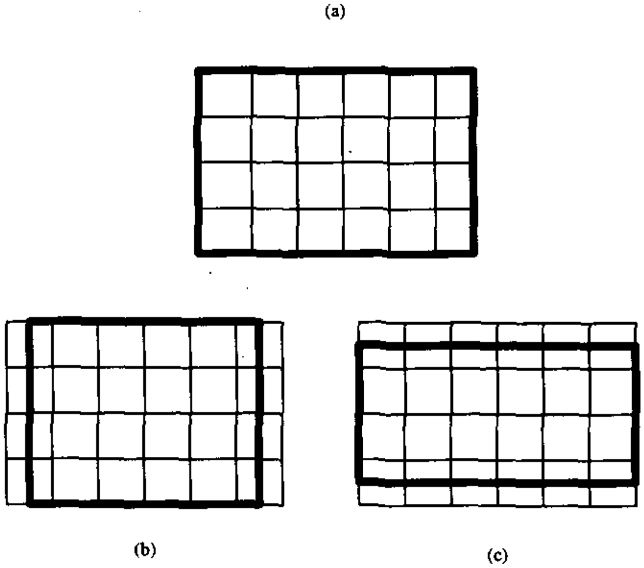

FIG. 6. Global conservation boundaries for the staggered grid (highlighted in bold): (a) Continuity equation; (b) u-momentum equation; (c) v-momentum equation.

overlap region. The global conservation boundary for the u-momentum equation is highlighted in bold. If in the overlap region, we sum over u control volumes in the upper block only, then the exclusive overlapping appears as shown in Fig. 7b. In Fig. 7b, numbers have been given to the various segments which will be used in the ensuing derivation of the conditions for global u-momentum conservation.

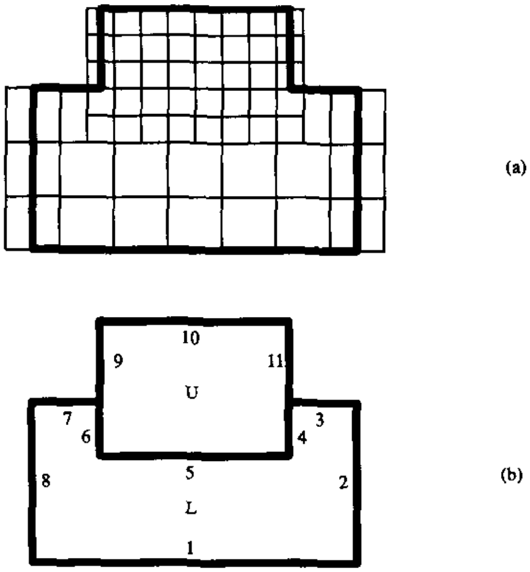

FIG. 7. *Composite grid constructed from two overlapping grid blocks:* (a) Composite grid with u-momentum conservation boundary highlighted in bold; (b) Exclusive overlapping used in derivation of global conservation conditions.

Summing over all the u control volumes and using the exclusive overlapping shown in Fig. 7b, we arrive at

$$\int_2 E\,dy - \int_8 E\,dy + \int_7 F\,dx + \int_3 F\,dx - \int_1 F\,dx$$
$$+ \int_{11} E\,dy - \int_9 E\,dy + \int_{10} F\,dx$$
$$= \left[\int_{6_U} E\,dy - \int_{6_L} E\,dy\right] + \left[\int_{4_L} E\,dy - \int_{4_U} E\,dy\right]$$
$$+ \left[\int_{5_U} F\,dx - \int_{5_L} F\,dx\right]. \tag{10}$$

The left-hand side of this expression is nothing but the summation of the boundary fluxes through the u-momentum global conservation boundary for the composite grid. The right-hand side of the expression represents the difference in the summation of the fluxes along the internal boundaries of the upper block as estimated from the two different blocks in the composite grid system. Now, for global conservation of u-momentum, the summation of the u-momentum flux through the u-momentum global conservation boundary should vanish. Accordingly, for global conservation of u-momentum, we arrive at

$$\int_{6_U} E\,dy = \int_{6_L} E\,dy, \qquad \int_{4_U} E\,dy = \int_{4_L} E\,dy,$$
$$\int_{5_U} F\,dx = \int_{5_L} F\,dx \tag{11}$$

which state that the u-momentum flux through any internal boundary must be identical when estimated from the blocks on either side of the internal boundary. Equations (11) state that for global conservation of u-momentum, no u-momentum can be created at internal boundaries. Similar conditions apply for the v-momentum and continuity equations.

Global Conservation Strategy

Because we are using a pressure correction technique for solving the governing equations, the global conservation procedure involves a different strategy than that used with density-based compressible flow techniques using non-staggered grids. This is due in part to the nature of the staggered grid system, but more specifically it is due to the type of boundary condition used for solving the pressure correction equation. Two types of boundary conditions in general can be used for the pressure correction equation. If the pressure is known at the boundary, then the value of the pressure correction at the boundary can be set to zero. If the pressure is not known at the boundary, then the velocity component normal to the boundary must be specified. Since, in general, the boundary pressure values are not known, especially for blocks which are located completely interior to the physical boundaries of the domain, we have adopted the use of normal velocity component boundary conditions in this work exclusively. Normal velocity boundary conditions, in fact, can be used quite generally throughout, even at boundaries where the normal velocity component is not initially known (such as internal boundaries and outflow boundaries), but which evolves to a known value as the sequential solution process converges. With the use of normal velocity boundary conditions for the pressure correction equation, the pressure field is only obtained to within an arbitrary constant. However, this represents no problem in terms of the solution technique, since the density is unaffected by the magnitude of the pressure, and for single grid solutions, if a pressure value is known at a certain point in the field, then the entire pressure field can be adjusted accordingly, after the solution has been obtained. However, for composite grids, in which the governing equations are solved independently within each block, in a block-to-block iterative fashion, each block in the system will generate a pressure field independently from those created in neighboring blocks. It is this characteristic that requires a different global conservation strategy than that used in density-based compressible flow solvers applied to composite grids.

Solution of the continuity (pressure correction) equation requires the specification of the mass flux into each of the pressure correction control volumes located along the boundaries of the grid. Similarly, the solution of the u and v-momentum equations requires the specification of the u and v-momentum fluxes into each of the respective u and v control volumes located along the boundaries of the grid. Each of these control volume fluxes may involve contributions from fluxes from specified external boundary condition segments or from fluxes entering through internal boundaries of the grid. For the case of internal boundaries, flux contributions must be calculated in such a way that the global conservation conditions previously outlined are satisfied. A discussion of the exact procedure for achieving this will be undertaken in the following section. This section deals with issues concerning the global conservation strategy that has been adopted for a pressure correction algorithm with a staggered grid system. Thus, our global conservation strategy is as follows. Explicit conservation of mass across the horizontal and vertical sides of the global conservation boundary for the pressure correction equation is used to determine the mass fluxes into each of the pressure correction control volumes along the boundaries of the grid block in question. Once the mass fluxes have been determined, then the normal component of the velocity profile along the horizontal and vertical sides of the grid block can be determined. From these velocity values, we can compute the part of the u-momentum fluxes into each of the u

control volumes along the vertical boundaries of the grid block which do not involve pressure (e.g., the term $\rho uu - \mu(\partial u/\partial x)$), as well as the part of the v-momentum fluxes into each of the v control volumes along the horizontal boundaries which do not involve pressure (e.g., the term $\rho vv - \mu(\partial v/\partial y)$). The pressure values required for calculating the rest of these fluxes are already specified via the staggered grid arrangement. Since the u-momentum fluxes into the u control volumes along the horizontal boundaries do not contain a pressure term (i.e., $\rho uv - \mu(\partial u/\partial y)$), they can be obtained via explicit conservation, as was done for the mass fluxes. Similarly, this can be done for the v-momentum fluxes into the v control volumes along the vertical boundaries. In this way, all the required boundary fluxes for the various control volumes along the boundaries of the staggered grid block in question are obtained.

For a converged solution to a composite grid problem, since the pressure fields within each of the blocks of the system have developed independently from the others and are only determined to within a constant within each grid block, the pressure fields within the various grid blocks must be corrected in some manner so that they are compatible with each other. In order to determine the compatibility constants for each block, a normal momentum balance is performed across some segment of each internal boundary interface, and the pressure in one of the blocks sharing that interface is adjusted so that normal momentum flux across that segment is identical when computed from either of the two blocks. For the general case in which the grid lines from neighboring blocks across the interface are discontinuous, this postprocessing procedure is non-unique and dependent upon the choice of the segment over which the normal momentum flux balance is performed.

Configurations with Internal Obstacles

Since the present interface procedure is designed to satisfy the conservation laws without artificially imposing the continuity of solution variables, extra care must be taken in ensuring that no spurious multiple solutions occur for configurations that contain solid obstacles. For the staggered grid arrangement adopted here, there is no need to specify boundary conditions for pressure; this basic merit remains the same for any multi-block configuration. However, for multiply connected domains, it is known that extra boundary constraints need to be prescribed, otherwise the solution may not be physically correct. An earlier study conducted by Shyy [15] has established this point clearly. By solving a channel flow with two outlet branches, besides the normal extrapolating outflow conditions, an additional regulatory mechanism is needed. In Shyy [15], it is the ratio of the mass flux between the two branches that helps the numerical computation reach a unique and physically correct solution. This extra condition is a mathematically correct procedure, and not a numerical artifact. For example, Milne-Thompson [16] has studied a potential flow in a branched canal. In that analysis, the stagnation point of the flowfield is given a priori, amounting to an equivalent specification of the mass split ratio as well.

In the present formulation, for configurations with internal obstacles, there is no need to explicitly assign any extra boundary conditions to facilitate the numerical computation. The problem arises from the fact that since one does not have to prescribe any pressure boundary conditions within each block, potentially, a pressure jump can occur across a block interface. This jump is physically incorrect and can result in multiple spurious solutions. To circumvent this potential pitfall, one can simply enforce a pressure continuity constraint at the interfaces shared by adjacent blocks. The practice devised here is that in the course of iteration, the pressure continuity at interfaces common to adjacent blocks is enforced via explicit conservation of the normal momentum through these interfaces. For cases with internal obstacles, individual grid blocks need this continuous enforcement of pressure continuity at internal interfaces to prevent unphysical pressure jumps from appearing. As can be observed in the test cases to be presented later, the present procedure can maintain a physically correct solution. One should emphasize that the present treatment of pressure is completely consistent with the spirit of the staggered grid; i.e., no artificial pressure boundary condition is used to affect the final solution; the pressure continuity is enforced strictly via a normal momentum balance between adjacent blocks to avoid creating a nonphysical flowfield.

Explicit Conservation Procedure

Suppose we wish to update the values of the dependent variables in one of the blocks of a composite grid. These calculations require that the fluxes of mass and momentum through the boundaries of the block into the various boundary control volumes be specified. According to the global conservation strategy previously outlined, explicit conservation of the mass and u-momentum flux through the horizontal boundaries and the mass and v-momentum flux through the vertical boundaries, provides all of the information required for calculating the fluxes into all of the boundary control volumes. In this section, we detail the procedure for explicitly conserving mass and u-momentum through horizontal internal boundaries. The same concepts apply for explicitly conserving mass and v-momentum through vertical internal boundaries.

Consider a composite grid comprised of two blocks with a horizontal overlap region. Figure 8 shows a blowup of part of the overlap region, where the upper block (whose dependent variables we wish to update) is drawn with dotted lines and the neighboring lower block is drawn with

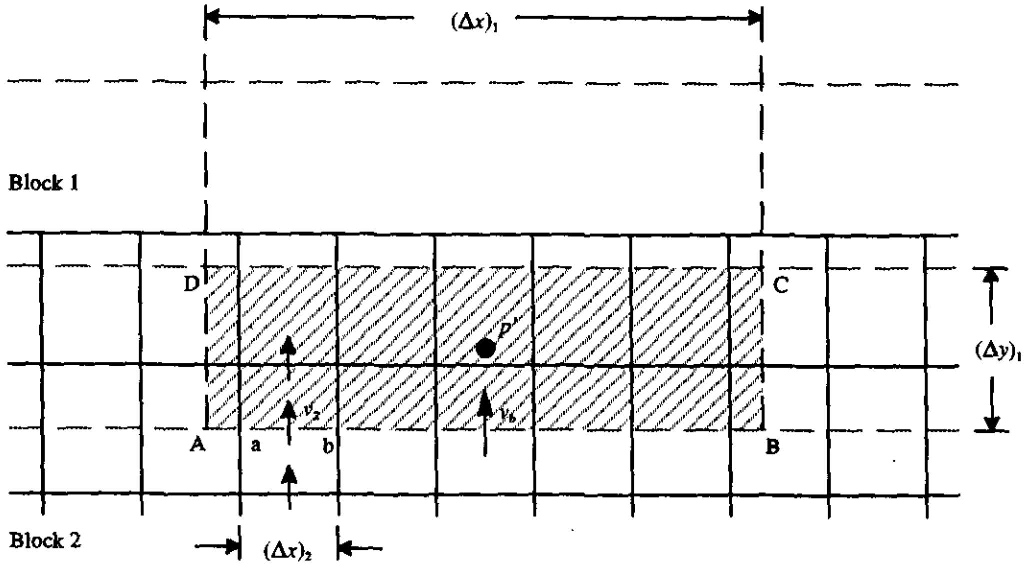

FIG. 8. Blowup of horizontal overlap region with a pressure correction control volume, *ABCD*, for block one highlighted.

solid lines. The upper block will hereafter be denoted as block one and the lower block as block two. In the figure, a pressure correction control volume along the lower boundary of block one has been shaded. In order to formulate the discrete form of the pressure correction equation for this control volume, the mass flux through the lower boundary of the control volume (segment AB) must be obtained. Since segment AB lies completely within block two, all the information required for calculating the mass flux through the segment is obtained from this block. Segment AB is comprised of a number of sub-segments, formed by the intersection of the vertical grid lines of block two with the segment. In this case, segment AB is comprised of five complete sub-segments (forming the center portion of the segment) and two partial segments (on the left and right ends of segment AB). Each of these sub-segments contributes a portion to the total mass flux through AB.

The mass flux contribution for a typical sub-segment in block two is obtained in the following manner. The constant normal velocity component along the segment is obtained using a linear distance-weighted interpolation from the velocity components just above and below, located within block two. For example, consider the second sub-segment, denoted as ab, of segment AB, shown in Fig. 8. The normal velocity component along the entire sub-segment is taken to be that at the center of the sub-segment, and is denoted v_2. The value of v_2 is obtained based on a linear interpolation within block two from the staggered grid values located directly above and below, which are also shown in the figure. Once the segment velocity has been found, the mass flux contribution is obtained by multiplying this value by the density and the segment length. This calculation procedure results in a piecewise constant mass flux distribution over the segment AB. A piecewise constant mass flux distribution allows the correct mass continuity to be recovered for an interface with continuous grid lines from the two adjacent blocks. With the total mass flux from block two denoted as $mflux_2$, and assuming that the north, west, and east pressure correction neighbors are internal unknowns of block one, then the discretized form of the pressure correction equation for the pressure correction control volume $ABCD$ becomes

$$a_P p'_P = a_E p'_E + a_W p'_W + a_N p'_N + b \tag{12a}$$

$$a_P = a_E + a_W + a_N, \tag{12b}$$

where the coefficients a_i are as given before (except for a_s which has been set to zero since we consider the mass flux obtained to be a known quantity), and the source term b takes the form

$$b = [(\rho u^*)_w - (\rho u^*)_e](\Delta y)_1 + mflux_2 - (\rho v^*)_n (\Delta x)_1. \tag{13}$$

Once the mass flux into the pressure correction control volume has been calculated, the velocity component at the control volume boundary for block one, denoted v_b in the figure, is given by

$$v_b = \frac{mflux_2}{\rho(\Delta x)_1}. \tag{14}$$

Thus, as previously stated, conservation of mass along the horizontal boundaries provides both the boundary conditions for each of the pressure correction control volumes along the boundary, as well as the normal velocity component profile along the boundary. Once this profile is determined, then all the information required for the calculation of the parts of the v-momentum fluxes not involving pressure (i.e., $\rho vv - \mu(\partial v/\partial y)$) is available.

The procedure for explicitly conserving the u-momentum fluxes into each of the u control volumes along horizontal boundaries is similar to that for explicitly conserving mass. Consider again, a composite grid comprised of two blocks with a horizontal overlap region. Figure 9 shows a blowup of part of the overlap region, where the upper block (whose dependent variables we wish to update) is again denoted as block one, and the lower neighboring block as block two. In the figure, a u control volume, labelled $ABCD$, along the lower boundary of the block has been shaded. In order to formulate the discrete form of the u-momentum equation for this control volume, the u-momentum flux through the lower boundary of the control volume (segment AB) must be obtained. Again, segment AB is comprised of a number of sub-segments; however, in this case the sub-segments are defined by the u control volumes in block two. Consider the sub-segment ab shown in the figure. The u-momentum flux, including the convective and diffusive components, through this sub-segment, denoted $uflux_{ab}$ is calculated as

$$uflux_{ab} = [D(1 - 0.5\,|F/D|) + [\![F, 0.0]\!]]\, u_L - [D(1 - 0.5\,|F/D|) + [\![-F, 0.0]\!]]\, u_U, \quad (15)$$

where the terms u_U and u_L are respectively the u velocity components located on the staggered grid of block two just above and below the segment. In this expression, the second-order central difference scheme has been used for both the convection and diffusion terms for illustration purposes. The terms F and D represent the convection and diffusion fluxes through the segment and are given as

$$D = \frac{\mu\, \Delta x}{(\Delta y)_2}, \qquad F = \rho v_s(\Delta x). \quad (16)$$

The term Δx represents the length of the sub-segment, and $(\Delta y)_2$ is the constant vertical grid spacing in block two. The quantity v_s is the velocity component calculated via bi-linear interpolation from the neighboring v component values of block two at the center of the segment. With the total flux contribution from block two denoted as $uflux_2$ and assuming that the north, east, and west neighbors of the u boundary control volume in block one are internal unknowns, the discretized form of the u-momentum equation for the control volume becomes

$$A_P u_P = A_E u_E + A_W u_W + A_N u_N - (F_n + F_e - F_w)\, u_P + (p_W - p_E)(\Delta y)_1 + uflux_2 \quad (17a)$$

$$A_P = A_E + A_W + A_N. \quad (17b)$$

In the original formulation for the u control volume shown in Eq. (3e), the term $(F_n - F_s + F_e - F_w)\, u_P$ is identically zero for a continuity satisfying velocity field and can be dropped. The term $(F_n + F_w - F_e)\, u_P$ remains in Eq. (17a) because A_P is now computed without the contribution from A_S. Due to the interface treatment, A_S and its corre-

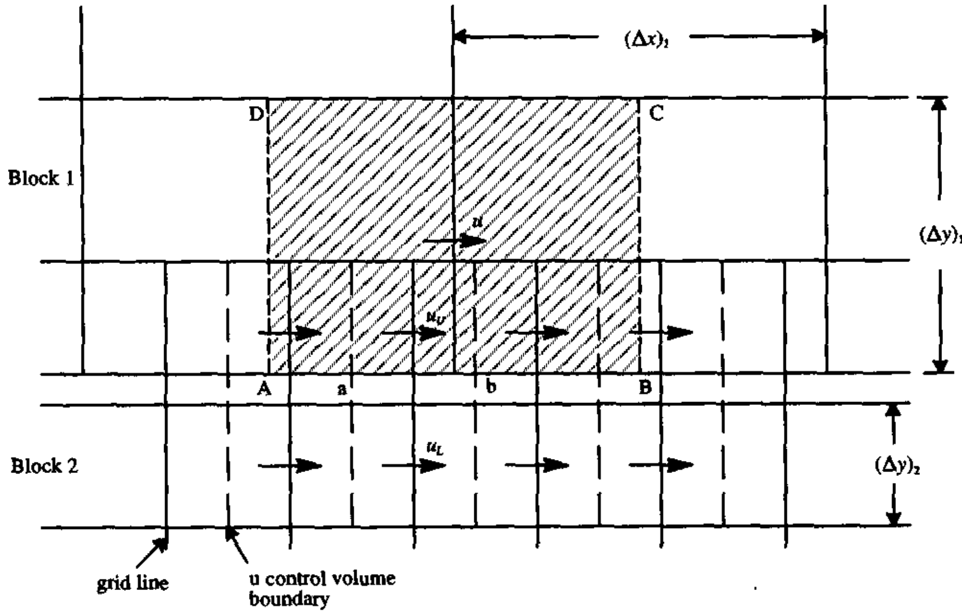

FIG. 9. Blowup of horizontal overlap region with a u control volume, $ABCD$, for block one highlighted.

sponding component in A_P is now explicitly given in the form of $uflux_2$. The term $(F_n + F_w - F_e)$ can be replaced by F_s and computed using the normal velocity component profile at the boundary obtained from explicitly conserving mass.

In the examples above, illustrating the explicit procedure for conserving mass and u-momentum across horizontal boundaries, all the required information was obtained from a single neighboring block. It should be noted here, that in a general composite grid, for any control volume located along a boundary, contributions to the total flux into the control volume may be required from any number of neighboring blocks and/or externally imposed boundary conditions. Since the path of information flow into each block is entirely specified via boundary segments, we can determine which neighboring block or boundary condition is to provide the required information across a particular portion of a control volume boundary. Once this is known, then if a neighboring block is required to provide the information, the flux through that portion can be calculated as described above; and if a boundary condition is required, then the flux across that portion can be calculated accordingly.

4. RESULTS

In order to demonstrate the capability of the composite grid procedure which has been developed, the results from three different cases will be presented. The first case is that of an n-shaped lid-driven cavity composed of three overlapping grid blocks. The next two cases involve two different multiply connected regions, one, a channel with two internal baffles, and the other a CFD configuration. All cases were computed using the second-order central difference scheme for both the convection and diffusion terms.

n-shaped Cavity

In this case, we compute the flow in a lid-driven n-shaped cavity for a Reynolds number of 2000 based on the cavity

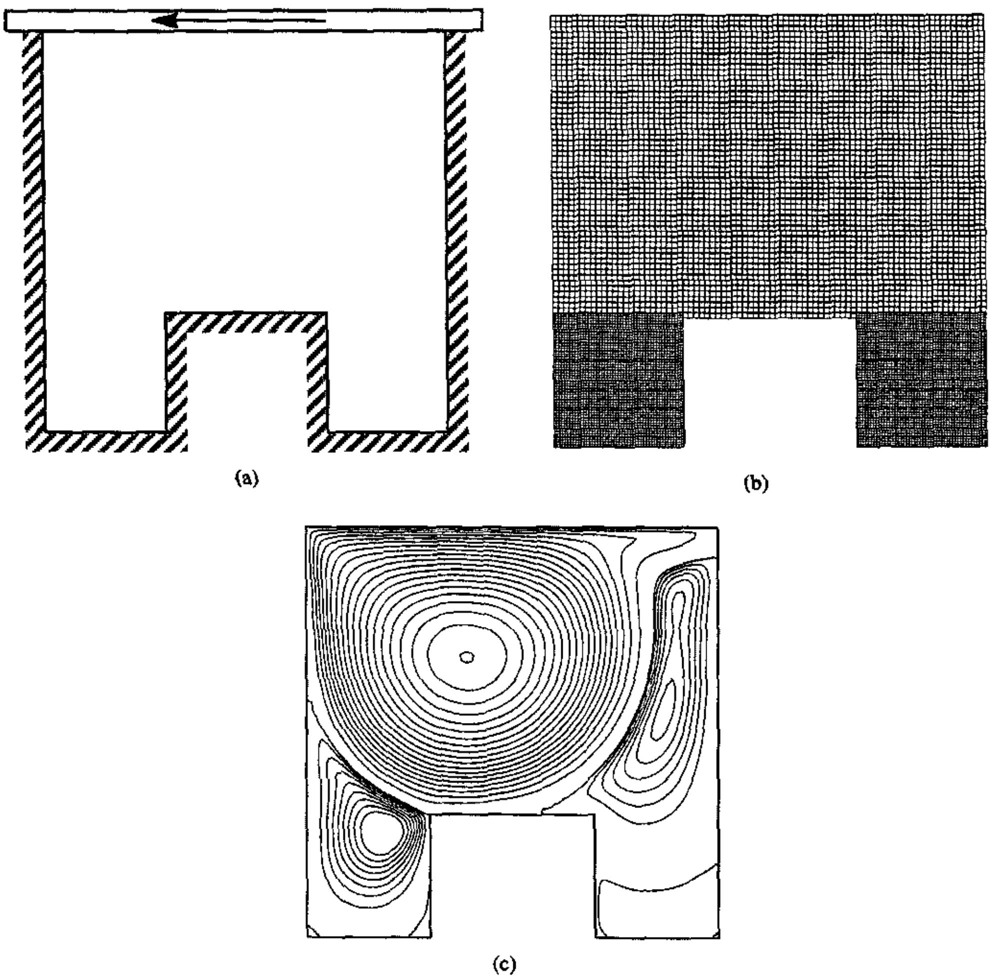

FIG. 10. n-shaped cavity: (a) Physical problem with boundary conditions; (b) Composite grid; (c) Stream function contours (Re = 2000).

width and lid velocity. The physical problem with boundary conditions is shown in Fig. 10a. The grid used is shown in Fig. 10b. Three grid blocks are used to cover the domain. Block one which covers the main body of the cavity has a resolution of 81 × 57. Blocks two and three, which cover the legs of the cavity both have a resolution of 49 × 51. The grid spacings for blocks two and three are exactly half of those for block one, in both directions. Blocks two and three overlap block one in horizontal strips which have a thickness of one row of coarse grid cells or two rows of fine grid cells. For this case, use of direct interpolation for supplying the required internal boundary information results in incompatible boundary conditions which prevent the algorithm from converging. This problem exists for all cases in which

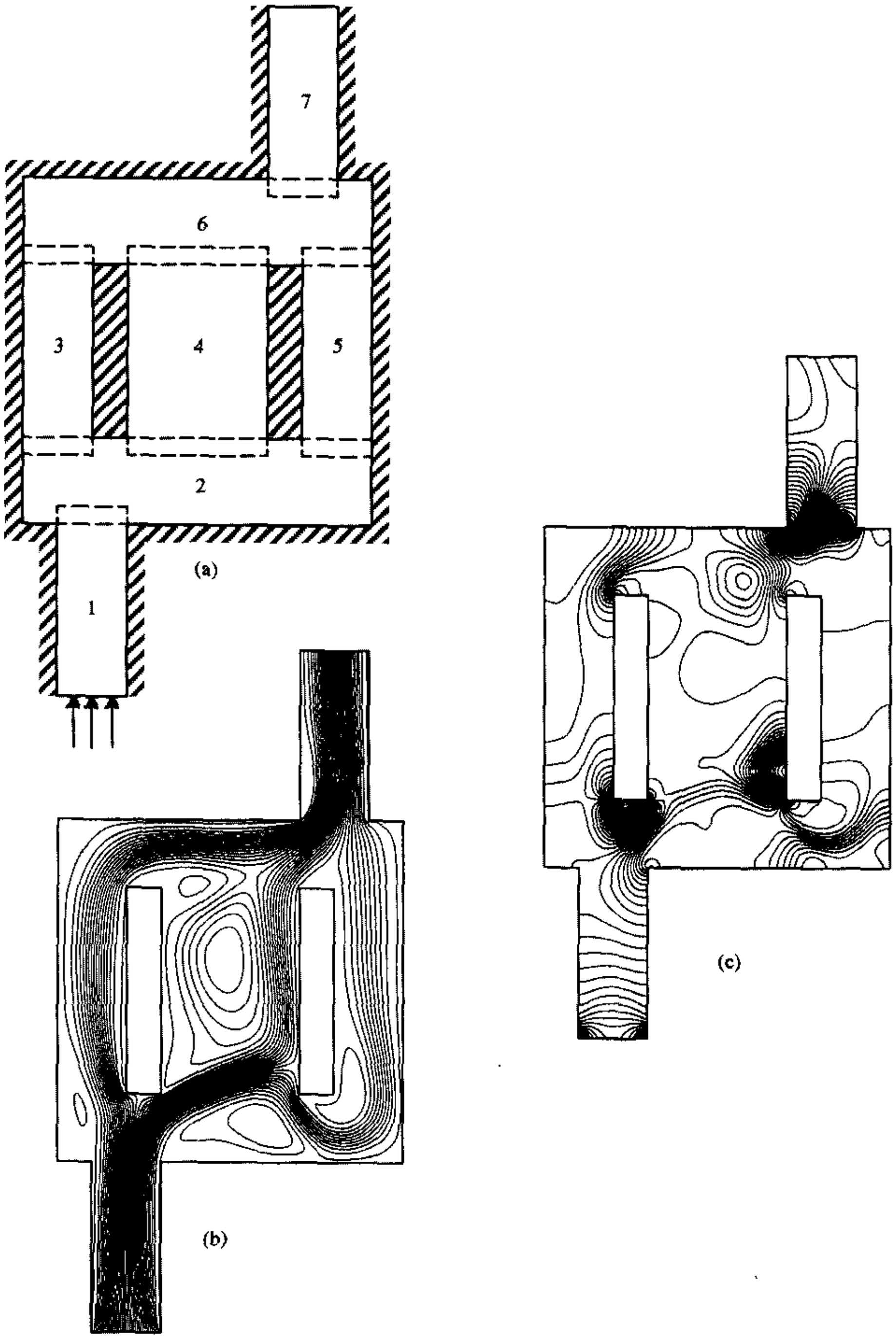

FIG. 11. Channel with baffles: (a) Physical problem and grid block arrangement with overlap regions indicated by dotted lines; (b) Stream function contours (Re = 800); (c) Pressure contours (Re = 800).

an interior block exists which does not contain an outflow boundary whose velocities can adjust to satisfy the global mass continuity constraint. With the interface treatment adopted herein; no such problem exists. Figure 10c displays the stream function contours obtained using the current procedure. A primary recirculation region exists within the main body of the cavity. Two secondary recirculation regions are seen associated with the primary recirculation, one whose center is located within the left leg of the cavity and the other whose center lies above the right leg of the cavity. Both secondary recirculation regions pass smoothly through the internal boundaries separating the fine grid blocks of the cavity legs from the coarse grid block of the main body of the cavity.

Channel with Baffles

In this case, we compute the flow through a multiply connected region, namely, a channel with internal baffles for a Reynolds number of 800 based on the configuration width. The physical configuration along with the block arrangement is shown in Fig. 11a. The composite grid used for this case consists of seven grid blocks, each having the same constant grid spacing. Block one has a resolution of 21×56, block two, 101×21, block three, 21×71, block four, 41×71, block five, 21×71, block six, 101×21, and block seven, 21×56. The blocks in the composite grid are connected by overlap regions that are five grid cells thick and which are shown in the figure with dotted lines. Streamlines of the flow are shown in Fig. 11b. Two fairly large recirculation regions exists on either side of the entrance into the channel. A large recirculation region also exists in the center of the middle passage of the channel, created due to the large flow deflection angle caused by the impingement of the main flow on the baffle located near the entrance. Some of the main flow which initially passed through the leftmost passage of the channel is drawn deep into the middle passage, encompassing the recirculation region there, before being drawn out by the main flow through the middle passage to the channel exit. Pressure contours are displayed in Fig. 11c. From this figure it is seen that the numerical computation yields a physically realistic solution with no pressure jumps across the internal boundaries separating the grid blocks, consistent with our expectation of the block interface treatment.

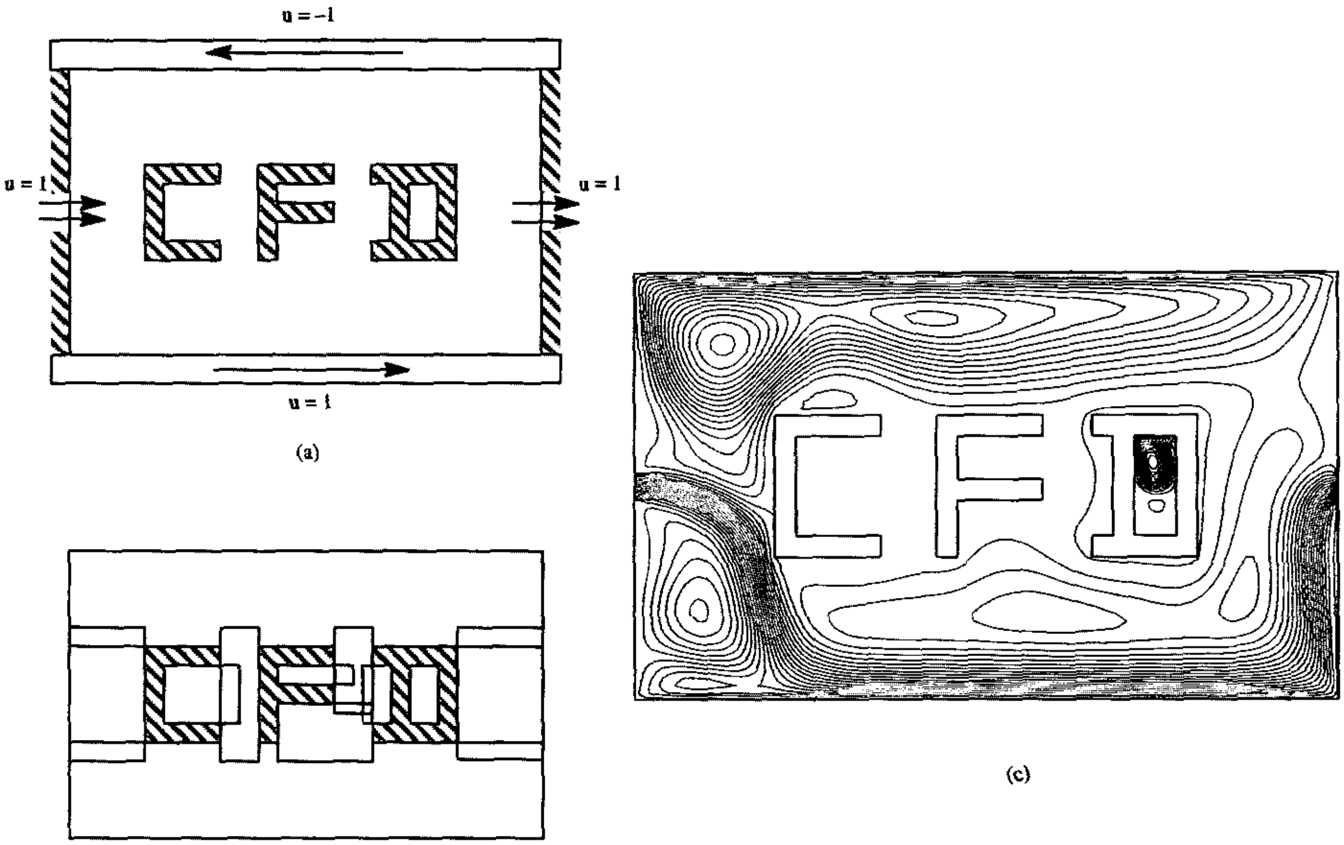

FIG. 12. CFD configuration: (a) Physical problem with boundary conditions; (b) Grid block arrangement; (c) Stream function contours (Re = 1000).

CFD Configuration

In this case, we demonstrate the ease with which the current procedure can handle problems with very complex geometry. Figure 12a shows the configuration studied, where we have the internal obstacle CFD inside a cavity with sliding upper and lower walls and entry and exit jets on the left and right walls, respectively. Figure 12b shows the multi-block configuration used to solve the problem. The grid consists of 11 individual grid blocks, 10 of which are used to form the interior portion of the cavity, and one which is used on the interior of the letter *D*. The problem prescribed on the block forming the interior of *D* is actually a separate lid-driven cavity flow running simultaneously with the problem prescribed in the main cavity. Both flows are computed for a Reynolds number of 1000, based on the lid velocity and cavity width of each problem. Figure 12c displays the streamfunction contours.

5. CONCLUDING REMARKS

In this study, a flexible composite grid procedure has been developed for solving the incompressible Navier–Stokes equations on domains composed of an arbitrary number of overlapping grid blocks. A pressure correction technique is used, along with a staggered grid system, to solve the governing equations independently within each of the blocks of the composite grid. A conservative internal boundary treatment is used for transferring information between the blocks. An organizational scheme has been presented which allows us to implement the conservative internal boundary scheme in a straightforward, systematic manner. Several examples have been given, showing the utility of the composite grid procedure in handling problems with complex geometry. The methodology developed herein can make a useful contribution in solving many problems with complex geometry.

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